GEOMETRY.

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85. Proposed by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.

Prove by pure geometry. Give direct proof, if possible.

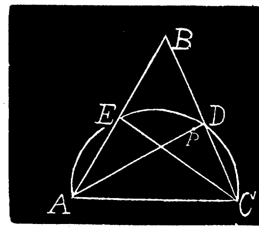
If the bisectors of two angles of a triangle are equal, the triangle is isosceles.

[From *Wentworth's Plane Geometry*, exercise 43, page 72.]

I. Solution by J. M. COLAW, A. M., Monterey, Va., and EDMUND FISH, Hillsboro, Ill.

In triangle ABC, let AD, CE, be the bisectors of angles A and C, and $AD=CE$. Then angle A=angle C, and triangle ABC is isosceles.

Suppose a circle passed through A, C, and E. It will also pass through D. If not, suppose that it cut AD in any point P short of D. Then arc $EB>$ arc PE, since $\angle ECA(=\angle DCE)>\angle PCE$.



Also, arc PE=arc PC, since $\angle EAP=\angle PAC$. Whence arc $AEP>$arc EPC.

$\therefore$ chord $BP>$chord CE. But by hypothesis, $AD=CE$. $\therefore$ $AP>AD$, which is absurd. In the same way it may be shown that the supposition that the circle, which passes through A, C, and E, cuts BD in any point P', beyond D, also leads to an absurdity. The circle must therefore pass through D. Hence $\angle EAD(=\frac{1}{2}\angle A)=\angle DCE(=\frac{1}{2}\angle C)$.

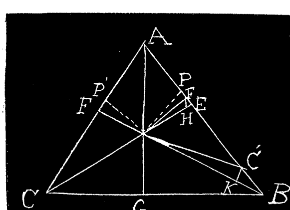
$\therefore$ $\angle A=\angle C$, and triangle ABC is isosceles.

II. Solution by OTTO CLAYTON, Teacher of Mathematics and Physics, Remington High School, Remington, Ind.

Draw the bisector AG meeting the given bisectors CE and BF of the given triangle ABC, in the point O. Revolve ACG about AG as an axis until AC coincides with AB. Then the point C will fall within the segment AB, on the point B, or without the segment AB, according as $\angle ACO$ is greater, equal to, or less than $\angle ABO$. And the point F will fall within AE, on the point E or within EB, according as $\angle ACO$ is greater, equal to, or less than $\angle ABO$.

But $\angle ACO$ cannot be greater than ABO, for $(C'O+OE)=CE$ would be less than $(BO+OF')=BF$, which is contrary to the hypothesis that $CE=BF$.

Likewise $\angle ACO$ can not be less than $\angle ABO$, for $(B'O+OE)=CE$ would be greater than $(BO+OF)=BF$, which is contrary to the hypothesis. Therefore $\angle ACO=\angle ABO$, and C falls upon B. Therefore the triangle is isosceles.



I think this is a simple proof and does not involve anything outside of the first book of Wentworth.

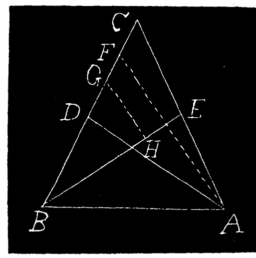
In the proof I did not show how $OE+OC'$ is less than $BO+OF'$. When the perpendicular OP falls between OE and OB the reason is obvious. When OP falls without, construct equilateral triangles $F'OH$ and $C'OK$. Then prove HE less than KB.

III. Solution by the PROPOSER.

Let the bisectors BE and AD be equal, prove triangle ABC isosceles.

Three suppositions are possible.

1st, $\angle A > \angle B$; 2nd, $\angle A < \angle B$; 3rd, $\angle A = \angle B$.



First, suppose $\angle A > \angle B$: then $\frac{1}{2}\angle A > \frac{1}{2}\angle B$. Construct $\angle FAD = \angle CBE$. Then in the triangle FAB, $FB > FA$ (greater side opposite greater angle). Lay off on BF a distance BG equal to AF, and draw GH parallel to FA. Then the triangle BGH=triangle FAD ($BG = FA$, by construction, $\angle GBH = \angle FAD$, for the same reason, $\angle BGH = \angle DFA$, exterior-interior angles.)

$\therefore$ $DH = BA$ (homologous sides of equal triangles) *which is absurd*, because $BE = AD$, by hypothesis, and BH is only a part of BE.

Second, in a similar manner it can be shown that $\angle B$ cannot be greater than A; *i. e.* $\angle A$ cannot be less than $\angle B$.

Third, as $\angle A$ can neither be greater nor less than $\angle B$, it must be equal to $\angle B$. $\therefore$ the triangle is isosceles. Q. E. D.

For other demonstrations of this problem, see Vol. II., pages 158, 189—192. EDITOR.

86. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Prove that the four conics which have S for focus and which touch the three sides of each of the triangles ABC, AEF, BFD, CDE, have their latera-recta equal.

Solution by the PROPOSER.

Reciprocate with respect to S; then we have the three altitudes of an equilateral triangle passing through a point, and the circumscribing circles of the four triangles formed by joining the feet of the perpendiculars equal.

The latera recta of the given conics are then equal.

87. Proposed by WALTER HUGH DRANE, A. M., Professor of Mathematics, Jefferson Military Academy, Washington, Miss.

Given any two straight lines in space, AB, CD, which do not intersect. So construct upon one of the lines as base, a triangle, having its vertex in the other line, such that its perimeter shall be a minimum.

No solution of this problem has been received.

88. Proposed by FREDERICK R. HONEY, Ph. B., Instructor in Mathematics in Trinity College, New Haven, Conn.

Prove that the volume of the frustum of a cone is equal to one-sixth of the altitude multiplied by the sum of the areas of the upper base, the lower base, and four times the area of the section midway between the upper and lower bases.

Solution by FREMONT CRANE, Sand Coulee, Mont.; ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa; G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va., and the PROPOSER.

Let R=radius of the lower base; r=radius of the upper base; ρ=radius